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Symmetrical Bending of Multicore Circular Sandwich Plates

FAKHRUDDIN ABDULHADI*
IBM General Systems Division, Rochester, Minn.

Introduction

SYMMETRICAL bending of single core circular plates was treated by Reissner,[1] Zaid,[2] and Erickson.[3] Unsymmetrical bending was treated by Kao.[4,5] The bending of multicore circular plates with membrane facings was treated by Stickney and Abdulhadi,[6] where the distinct cores were assumed to possess a common shear angle, in a manner similar to that of Liaw and Little[7] for rectangular plates. Kao and Ross[8] have shown, for multicore beams, that the customarily used common shear angle restriction introduces errors when the cores are not identical.

Analysis of multicore plates suitable for thin or thick cores and without the common shear angle restriction, to our knowledge, does not exist. This Note treats the symmetrical bending for such plates. The governing equations for five-layer plates are derived and solved for arbitrary symmetrical loading and boundary conditions.

Analysis

The facings are distinct, homogeneous and isotropic. The bending and extensional stiffnesses of each facing is taken into account. The facings carry plane stresses while the isotropic cores carry transverse shear stress only. The thickness does not change and the layers do not slip when the plate is loaded.

Let the odd subscripts $i = 1,3,5$ refer to bending layers (facings) and even subscripts $j = 2,4$ refer to shear layers (cores) as shown (Fig. 1). The forces and moments per unit length acting on ith layer are defined as

$$[N_{ir},N_{i\theta}] = \int[\sigma_{ir},\sigma_{i\theta}]dz_i \qquad [M_{ir},M_{i\theta}] = \int[\sigma_{ir},\sigma_{i\theta}]z_i dz_i \quad (1)$$

where $N_{ir}(M_{ir}),N_{i\theta}(M_{i\theta})$ are radial and tangential forces (moments), respectively, and $\sigma_{ir}(\sigma_{i\theta})$ is the radial (tangential) stress. The distance z_i is measured transversely from the center of ith facing.

Hooke's law is used in Eq. (1) leading to

$$\begin{bmatrix} N_{ir} \\ N_{i\theta} \\ M_{ir} \\ M_{i\theta} \end{bmatrix} = \begin{bmatrix} 1 & \nu & 0 & 0 \\ \nu & 1 & 0 & 0 \\ 0 & 0 & -1 & -\nu \\ 0 & 0 & -\nu & -1 \end{bmatrix} \begin{bmatrix} k_i u_{i,r} \\ k_i u_i/r \\ D_i w_{,rr} \\ D_i w_{,r}/r \end{bmatrix} \quad (2)$$

$$k_i = E_iT_i/(1-\nu^2) \qquad D_i = E_iT_i^3/[12(1-\nu^2)] \quad (2a)$$

where ν is common Poisson's ratio, $u_i(w)$ is radial (transverse) displacement, E_i is elastic modulus, and T_i is thickness. Summing moments about the middle of layer five, leads to resultant moment equations

$$M_r = -D(w_{,rr} + \nu w_{,r}/r) - \sum_1^3 \delta_{i5}k_i(u_{i,r} + \nu u_i/r) \quad (3)$$

$$M_\theta = -D(\nu w_{,rr} + w_{,r}/r) - \sum_1^3 \delta_{i5}k_i(\nu u_{i,r} + u_i/r) \quad (4)$$

where $D = D_1 + D_3 + D_5$, and δ_{i5} is the distance between centers of layers five and i. Since bending only is considered, the resultant forces vanish; that is

$$N_{1r} + N_{3r} + N_{5r} = 0 \qquad N_{1\theta} + N_{3\theta} + N_{5\theta} = 0 \quad (5)$$

The equilibrium equations of a differential plate element and Eqs. (3) and (4) are used to obtain an expression for the shear force Q_r

$$Q_r = -D(\nabla^2 w)_{,r} - \sum_1^3 \delta_{i5}k_i S(u_i)$$
$$\nabla^2(\ ) = (\ )_{,rr} + (\ )_{,r}/r \qquad S(\ ) = \nabla^2(\ ) - (\ )/r^2 \quad (6)$$

The core shear stresses τ_{2rz} and τ_{4rz} are obtained from the equilibrium equations of each facing and Eq. (2)

$$\tau_{2rz} = -k_i S(u_i) \qquad \tau_{4rz} = -\sum_1^3 k_i S(u_i) \quad (7)$$

The no-slip conditions between the various layers are

$$u_3 - u_1 = -\delta_{13}w_{,r} + C_2\tau_{2rz} \quad (8)$$

$$u_5 - u_3 = -\delta_{35}w_{,r} + C_4\tau_{4rz} \quad (9)$$

where $C_j = T_j/G_j$ and G_j is shear modulus of jth core. The displacement u_5 is related to u_1 and u_3 as

$$k_5u_5 = -(k_1u_1 + k_3u_3) \quad (10)$$

A set of three governing equations are obtained by use of the equilibrium equations of a plate element and Eqs. (6–10).

$$(1/r)(d/dr)rS(Dw_{,r} + \delta_{15}k_1u_1 + \delta_{35}k_3u_3) = p(r) \quad (11)$$

$$\delta_{13}w_{,r} + C_2k_1S(u_1) - u_1 + u_3 = 0 \quad (12)$$

$$\delta_{35}k_5w_{,r} + (C_4k_1k_5S - k_1)u_1 + (C_4T_3k_5S - k_3 - k_5)u_3 = 0 \quad (13)$$

where $p(r)$ is an arbitrary symmetric load per unit area.

Equations (11–13) are uncoupled to eliminate u_3 and reduce the number of equations to two. The uncoupling process leads to

$$e\frac{dw}{dr} = -fu_1 + b\frac{d}{dr}\frac{1}{r}\frac{d}{dr}(ru_1) + F(p,r) + \frac{A_1r}{2}\left(\log r - \frac{1}{2}\right) + \frac{A_2r}{2} + \frac{A_3}{r} \quad (14)$$

$$\left(\nabla^2 - \frac{1}{r^2} - \lambda^2\right)\left(\nabla^2 - \frac{1}{r^2} - \beta^2\right)u_1 = \left[\alpha F(p,r) - \frac{\eta}{r}\int rpdr\right] + (\alpha A_3 - \eta A_1)\frac{1}{r} + \frac{\alpha A_1 r}{2}\left(\log r - \frac{1}{2}\right) + \frac{\alpha A_2 r}{2} \quad (15)$$

where

$$e = D - \delta_{13}\delta_{35}k_3 \qquad f = \delta_{15}k_1 + \delta_{35}k_3 \quad (15a)$$

$$\alpha = (\delta_{15}k_1 + \delta_{13}k_3)/DC_2C_4k_1k_3k_5 \qquad \eta = \delta_{13}/C_2k_1D \quad (15b)$$

$$F(p,r) = \frac{1}{r}\int rdr\int\frac{dr}{r}\int rpdr \qquad b = \delta_{35}C_2k_1k_3 \quad (15c)$$

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* Staff Engineer, Acoustics Laboratory.

$$\lambda^2 = \frac{A}{2} - \frac{1}{2}(A^2 - 4B)^{1/2}$$
$$\beta^2 = \frac{A}{2} + \frac{1}{2}(A^2 - 4B)^{1/2} \quad (15d)$$

$$A = \frac{C_4Dk_5(k_1 + k_3) + C_2Dk_1(k_3 + k_5) + k_1k_3k_5(C_4\delta_{13}^2 + C_2\delta_{35}^2)}{DC_2C_3k_1k_3k_5} \quad (15e)$$

$$B = [D(k_1 + k_3 + k_5) + \delta_{15}^2k_1k_5 + \delta_{35}^2k_3k_5 + \delta_{13}^2k_1k_3]/DC_2C_4k_1k_3k_5 \quad (15f)$$

The complete solutions of Eqs. (14) and (15) are

$$u_1 = \frac{H_1}{r} + H_2r + H_3r\log r + \phi_2 + A_4I_1(x) + A_5I_1(y) + A_7K_1(y) \quad (16)$$

$$w = \frac{1}{e}\int(F - f\phi_2)dr + \frac{b}{e}\frac{1}{r}\frac{d}{dr}(r\phi_2) + L_1 + L_2\log r + L_3r^2 + L_4r^2\log r + L_5I_0(x) + L_6K_0(x) + L_7I_0(y) + L_8K_0(y) \quad (17)$$

where

$$x = \lambda r; \; y = \beta r \quad (17a)$$

$$H_1 = \alpha A_3/\lambda^2\beta^2 + (A_1/\lambda^2\beta^2)[\alpha(\lambda^2 + \beta^2)/\lambda^2\beta^2 - \eta] \quad (17b)$$

$$H_2 = \alpha(2A_2 - A_1)/4\lambda^2\beta^2 \qquad H_3 = \alpha A_1/2\lambda^2\beta^2 \quad (17c)$$

$$L_1 = b(2H_2 + H_3)/e + A_8 \qquad L_2 = (A_3 - fH_1 + 2bH_3)/e \quad (17d)$$

$$L_3 = (A_2 - A_1 + fH_3 - 2fH_2)/4e$$
$$L_4 = (A_1 - 2fH_3)/4e \quad (17e)$$

$$L_5 = A_4(b\lambda^2 - f)/e\lambda \qquad L_6 = A_5(-b\lambda^2 + f)/e\lambda \quad (17f)$$

$$L_7 = A_6(b\beta^2 - f)/e\beta \qquad L_8 = A_7(-b\beta^2 + f)/e\beta \quad (17g)$$

The terms I and K are the modified Bessel functions of the first and second kind, respectively. The function ϕ_2 is the particular solution of Eq. (15) due to the terms between the brackets. This particular solution can be found by the power series method when the load $p(r)$ is specified.

When a concentric hole is present, the unknown constants are $A_1, \ldots, A_8$. If no hole is present, the various displacements and moments must be finite at the center, that is, $A_1 = A_3 = A_5 = A_7 = 0$ and, consequently, $H_1 = H_3 = L_2 = L_4 = L_6 = L_8 = 0$. All moments, forces, stresses, strains, u_3, and u_5 may be obtained since they are all expressed in terms of u_1, w, and their derivatives. The boundary conditions at an edge $r = R$ for various supports are given as:

a) Simply supported edge

$$w = 0 \quad (18)$$

$$u_{1,r} + \nu u_1/r = 0 \qquad \left(\frac{d}{dr} + \frac{\nu}{r}\right)S(u_1) = 0 \quad (19)$$

$$\left(\frac{d}{dr} + \frac{\nu}{r}\right)F(p,r) + \frac{1+\nu}{2}(A_1\log r + A_2) + \frac{1-\nu}{4r^2}(A_1r^2 - 4A_3) = 0 \quad (20)$$

b) Free edge, Eqs. (19), (20), and $A_1 + \int prdr = 0$.

c) Clamped edge

$$w = 0; \; u_1 = 0; \; S(u_1) = 0 \quad (21)$$

$$F(p,r) + (A_1r/2)(\log r - \tfrac{1}{2}) + A_2r/2 + A_3/r = 0 \quad (22)$$

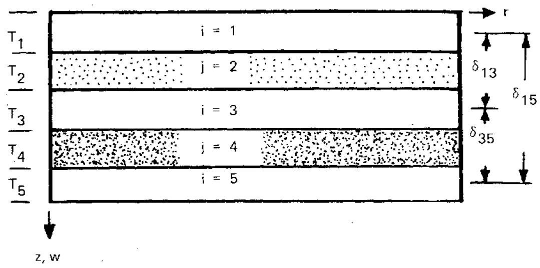

Fig. 1 Section of plate element.

Example

A simply supported plate of radius R is subject to the uniform load $p(r) = q$. The particular solution ϕ_2 is found by the power series method as

$$\phi_2 = H_4r + H_5r^3 \quad (23)$$

$$H_4 = -\eta q/2\lambda^2\beta^2 + \alpha q(\lambda^2 + \beta^2)/2\lambda^4\beta^4$$
$$H_5 = \alpha q/16\lambda^2\beta^2 \quad (23a)$$

Using Eqs. (16–20), (23), the given load q, and the fact that there is no hole in the plate leads to the complete solution for the displacements

$$u_1 = (H_2 + H_4)r + H_5r^3 + A_4I_1(x) + A_6I_1(y) \quad (24)$$

$$w = L_1 + \frac{2bH_4}{e} + \frac{r^2}{2e}(2eL_3 - fH_4 + 8bH_5) + \frac{r^4}{64e}(q - 16fH_5) + L_5I_0(x) + L_7I_0(y) \quad (25)$$

where

$$x_0 = \lambda R \qquad y_0 = \beta R \quad (25a)$$

$$A_2 = \frac{-(3+\nu)qR^2}{8(1+\nu)} \qquad A_4 = \frac{-\beta^3\mathcal{L}A_6 - 8(1+\nu)H_5}{\lambda^3\psi} \quad (25b)$$

$$A_6 = \frac{-8(1+\nu)H_5 + \lambda^2(3+\nu)H_5R^2 + \lambda^2(1+\nu)(H_2 + H_4)}{\mathcal{L}\beta(\beta^2 - \lambda^2)} \quad (25c)$$

$$A_8 = \frac{b\alpha A_2}{e\lambda^2\beta^2} - \frac{2bH_4}{e} + \frac{R^2}{4e}\left[2fH_4 - 16bH_5 + A_2\left(\frac{\alpha f}{\lambda^2\beta^2} - 1\right)\right] + \frac{R^4}{64e}(16fH_5 - q) + \frac{A_4}{e\lambda}(f - b\lambda^2)I_0(x_0) + \frac{A_6}{e\beta}(f - b\beta^2)I_0(y_0) \quad (25d)$$

$$\psi = I_0(x_0) - \frac{1}{x_0}(1-\nu)I_1(x_0)$$
$$\mathcal{L} = I_0(y_0) - \frac{1}{y_0}(1-\nu)I_1(y_0) \quad (25e)$$

Conclusions

The system of equations derived may be easily applied for the symmetrical bending of circular sandwich plates of various boundary conditions. Once the particular solution ϕ_2 is obtained for the specified load, the evaluation of the unknown constants becomes a simple algebraic operation.

The same procedure may be followed for the treatment of a sandwich construction containing more than five layers. However, the number of governing equations will increase depending on the number of additional layers. When a large enough number of layers are present, an equivalent homogeneous analysis would be adequate to calculate the deflection pattern. Hence, the technique developed in this Note is of primary importance when the number of layers is relatively small. This analysis can be easily reduced to treat plates with two distinct facings and a single core. This analysis is applicable for sandwich plates with thick or thin cores provided the facings are not so thick as to introduce appreciable transverse shear deformation in addition to that of the cores.

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Transonic Flows by Coordinate Transformation

P. O. BARONTI,* S. ELZWEIG,† AND R. VAGLIO-LAURIN‡
Advanced Technology Laboratories Inc., Jericho, N. Y.

NUMERICAL solutions of transonic flows about two dimensional airfoils have been obtained by Murman and Cole[1] by integration of the transonic small disturbance equation in a finite domain around the airfoil. The far field boundary conditions are derived analytically but they have to be periodically recalculated during the computation.

This problem is avoided if the infinite domain around the airfoil is transformed into a finite one by a transformation of the independent variables x,y to the variables suggested by Sills,[2] e.g., the variables $\xi = \tanh \alpha x$ and $\eta = 1 - e^{-\beta y}$.

In the new coordinate system, the domain being defined by $-1 \leq \xi \leq 1$ and $0 \leq \eta \leq 1$, the small disturbance equation becomes

$$\alpha^2(1 - \xi^2)[(1 - \xi^2)\phi_{\xi\xi} - 2\xi\phi_\xi][(1 - M_\infty{}^2) - (\gamma + 1)\alpha M_\infty{}^2(1 - \xi^2)\phi_\xi] + \beta^2(1 - \eta)[(1 - \eta) \times \phi_{\eta\eta} - \phi_\eta] = 0 \quad (1)$$

and the exact boundary condition on the boundaries $\xi = \pm 1$ and $\eta = 1$ is $\phi = 0$. Along the axis, $\eta = 0$, the boundary condition is given, e.g., by $\phi_\eta = 0$, fore and aft of the airfoil, and by $\phi_\eta = F'(\xi)/\beta$ along the airfoil; $F(\xi)$ being the airfoil shape. Thus, the numerical process is reduced to seeking a relaxation solution in the finite domain, without the need of computing the far field values.

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* Senior Research Scientist. Member AIAA.
† Research Scientist.
‡ Consultant; also Professor of Aeronautics and Astronautics, New York University, New York. Associate Fellow AIAA.

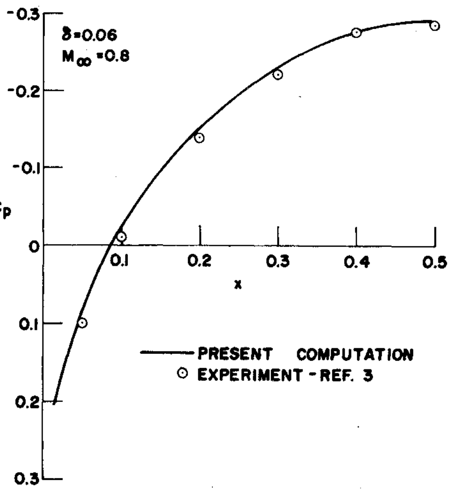

Fig. 1 Pressure coefficient along nonlifting circular arc airfoil.

A typical numerical solution of Eq. (1) for a nonlifting circular arc airfoil of thickness $\delta = 0.06$, at a freestream Mach number of 0.8, is presented here for illustrative purposes. The calculation employs a coarse mesh $\Delta\eta \simeq \Delta\xi = 0.05$ with $\alpha = \beta = 1$ and a point relaxation technique; the result is shown in Fig. 1 and compared with the experimental data of Knechtel.[3]

The computation has been repeated by using the approach of Murman and Cole. For comparable accuracy it has been found that the transformation affords considerable savings in computational time.

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Aerodynamic Characteristics of Slender Wedge-Wings in Hypersonic Strong Interaction Flows

C. M. RODKIEWICZ* AND T. K. CHATTOPADHYAY†
University of Alberta, Edmonton, Canada

THE purpose of this Note is to apply the results obtained by the present authors in Ref. 1 to predict the aerodynamic characteristics of slender two-dimensional wedge-wings in hypersonic strong interaction flow. In Ref. 1 the hypersonic strong-interaction flow over an inclined surface was analyzed using an asymptotic expansion in inverse powers of the inter-

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* Faculty Adviser, Department of Mechanical Engineering.
† Graduate Student, Department of Mechanical Engineering.